Definition of Terms

Principal (P) – original amount invested; amount upon which interest is based
Interest (I) – time value of money, cost of borrowing money over time
Future Value (FV) – principal plus interest = PV * IAF
Present Value (PV) – can be thought of as the principal; the principal that would result in a given future value = FV / IAF
Interest Adjustment Factor (IAF) – the factor which converts principal between present and future values.
Simple Interest – interest without compounding – IAF = (1+rn)
Compound Interest – receive or pay interest on interest – IAF = $(1+r)^n$
Ordinary Interest – interest based upon a 30 day month and a 360 day year
Nominal Interest – the stated, or so called rate. Nominal means in name only. Usually a rate which sounds better than it really is. Compare to APR.
Continuous compounding – theoretical maximum interest due to compounding – IAF = e^r
Effective Annual Rate – compound interest converted to an equivalent simple interest. EAR = IAF - 1
Add-on Interest – the use of simple interest to compute the total interest due on a loan
Annual Percentage Rate (APR) – the real interest rate which would amortize a loan as opposed to the nominal rate.
Amortization – the process of paying off a loan in fixed equal payments where interest is only applied to the unpaid balance.
Annuity – sequence of equal payments made at equal intervals of time
Savings Annuity – monthly savings plan. Monthly payment is denoted as “s”
Income Annuity – draw monthly payments from an initial amount. Monthly payment is denoted by “a”
Ordinary Annuity – annuity with payments made at the end of the period
Annuity Due – Annuity with payments made at the beginning of the period
Amortized Loan – Loan payments where interest is computed on the unpaid balance. Monthly payment is denoted as “m”
Points – a front end charge for obtaining a loan. It is really a front loaded interest charge. One point is 1%.
PITI – Letters to denote a complete mortgage payment amount consisting of principal, interest, taxes, and insurance

Simple Interest - interest without compounding

FV = PV + r PV
FV = PV (1+r), for 1 period

FV = PV (1 + nr), for n periods

Ordinary Interest – interest based upon 30 day months and a 360 day year

Compound Interest – interest upon interest

FV1 = PV (1+r)
$FV_2 = FV_1 (1+r) = PV (1+r) (1+r) = PV (1+r)^2$
$FV_n = FV_{n-1} (1+r) = PV (1+r)^n$

$FV = PV (1+r)^n$

Multiple Compounding Periods per Year

$FV = PV (1+ r/n)^n$, where n is the number of compounding periods in a year.

Continuous interest

The interest adjustment factor for continuous compounding is given by

$$\lim_{n\to\infty}\left(1+\frac{r}{n}\right)^n = e^r$$

and the continuous interest rate is e^r -1. Note that the IAF for multiple years with continuous compounding is e^{nr}. For example, to compare monthly compounding for five years with continuous compound the two formulas would be

$$\left(1+\frac{r}{12}\right)^{60} \text{ and } e^{5r}$$

Interest Rate Adjustment Factors (IAF) for n years compounded monthly

Simple	Compound	Continuous
(1 + nr)	$\left(1+\frac{r}{12}\right)^{12n}$	e^{nr}

Effective Annual Rate (EAR)

EAR = IAF - 1

Growth Rate- interest rate unknown

Given PV, FV, and n, find the interest rate that would have increase the PV to the FV.

$FV = PV\,(1+r)^n$
$FV/PV = (1+r)^n$
$(FV/PV)^{1/n} = 1+r$

$$\boxed{r = \left(\frac{FV}{PV}\right)^{\frac{1}{n}} - 1}$$

Growth Period – number of periods unknown

Given PV, FV, and r, find the number of periods required to increase PV to FV.

$FV = PV\,(1+r)^n$
$\log\,(FV/PV) = n \log\,(1+r)$

$$\boxed{n = \frac{\log(FV / PV)}{\log(1+r)}}$$

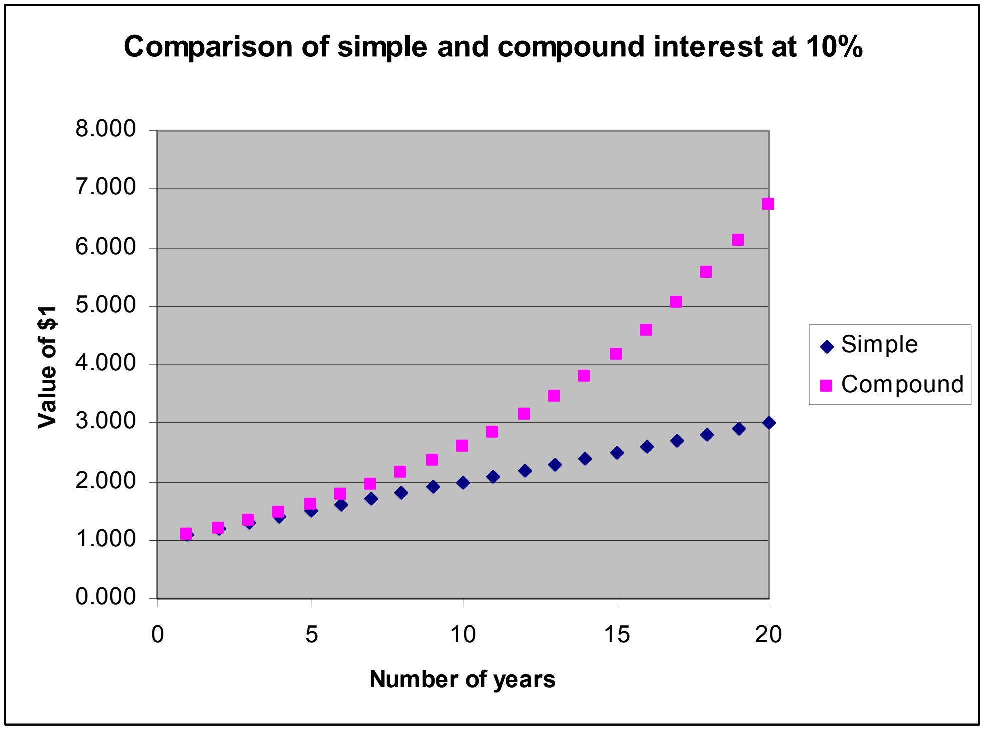
This can also be solved by trial and error with a calculator.

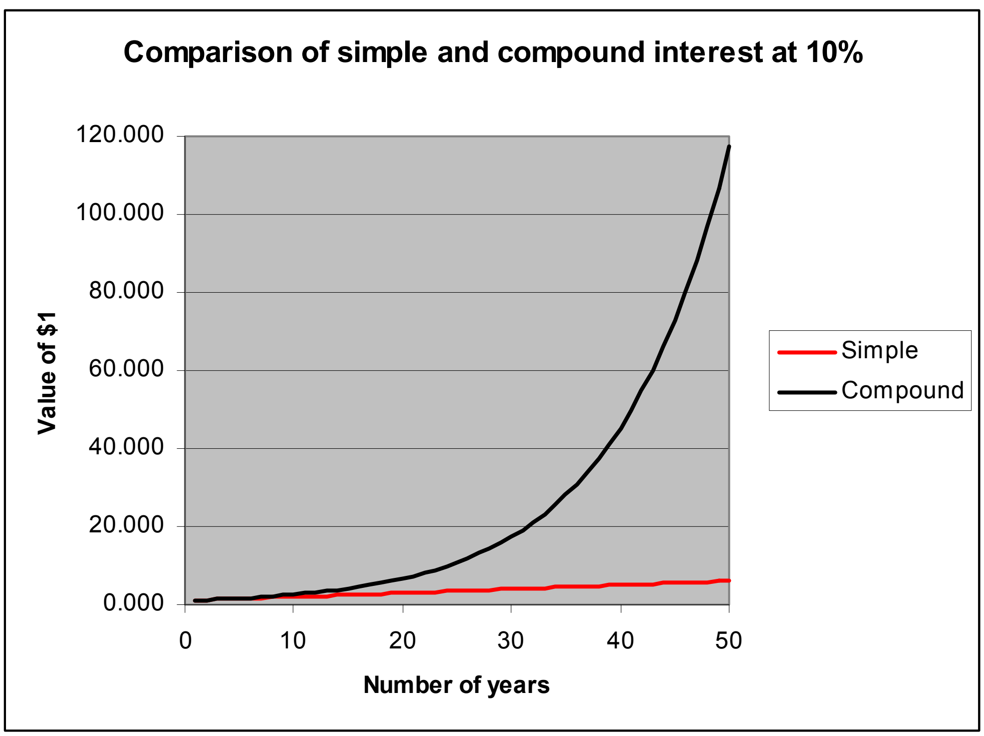
Doubling Period

Find the number of years required for a PV to double in value; that is, FV/PV = 2.

$$\boxed{n = \frac{\log 2}{\log(1+r)}}$$

Comparison of Simple and Compound Interest Growth





Savings Annuities

Annuities are sequences of equal payments made at fixed intervals (periods) of time. Annuity is confusing term because it is used in several different ways. There can be savings annuities, income annuities, and loan payment annuities. Right now we are going to restrict our discussion to savings annuities and will just call them annuities. Annuities made at the end of the periods are called ordinary annuities, and payments made at the beginning of periods are called annuities due. Annuities can be thought of as a monthly or period savings plan. Consider the following diagram of a $100 annuity made at the end of each year for five years.

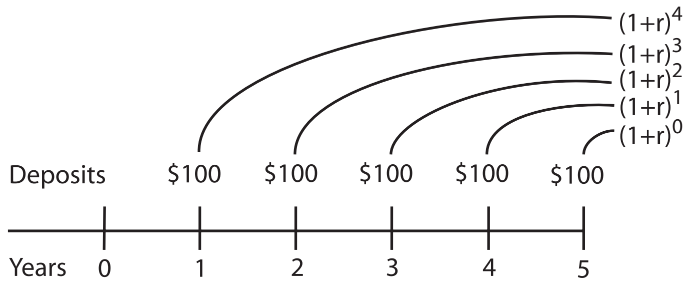


Figure 3. Annuity of $100 for five years.

The future value of this annuity can be written as the sum of five terms

$$FV = 100(1+r)^0 + 100(1+r)^1 + 100(1+r)^2 + 100(1+r)^3 + 100(1+r)^4$$

$$FV = 100\sum_{i=0}^{4}(1+r)^i$$

Before continuing, we will pause at this point and develop a general formula for solving a geometric progression like the one above. If we will let x take the place of (1+r) and s take the place of 100, then we have for an annuity of n periods

$$FV_n = \sum_{i=0}^{n-1} sx^i = s + sx + sx^2 + \cdots + sx^{n-1}$$

Now if we multiply FV_n by x we have

$$xFV_n = sx + sx^2 + \cdots + sx^{n-1} + sx^n$$

The sum XAn is nearly the same as An, and if they are subtracted from one another most of the terms cancel and we are left with

$$xFV_n - FV_n = FV_n(x-1) = sx^n - s = s(x^n - 1)$$

Which gives rise to the equation

$$FV_n = s\frac{x^n - 1}{x - 1}$$

but x = 1+r, and we can drop the n from FV and write

$$\boxed{FV = s\frac{(1+r)^n - 1}{r}} \quad \text{or} \quad \boxed{s = \frac{rFV}{(1+r)^n - 1}} \qquad \text{(Savings Annuity)}$$

If the annuity is an "annuity due", with the payment made at the beginning of the period, then the equation becomes

$$\boxed{FV = s(1+r)\frac{(1+r)^n - 1}{r}}$$

Notice that in the annuity due formula the FV is shift by one period or a multiple of (1+r). That is, everything earns interest for one more period.

Loans

When you borrow money you pay interest for the privilege of using some else's money. Interest allows you to buy something today and pay for it over time. It allows for instant gratification.

Interest on loans is usually calculated in one of two ways: add-on interest and amortized interest. With add-on interest you pay interest on the original amount of the loan every month even though you are gradually paying off the loan. With amortized interest you are only paying interest on the unpaid balance of the loan. Amortized interest is what you always want. Amortized interest quoted at the same rate as add-on interest is always a better deal than add-on interest. Add-on interest converted to an equivalent amortized interest rate is called annual percentage rate or APR. APR is the true interest, the interest rate you are really paying.

Add-on Interest

About the only advantage add-on interest has is that it is easy to calculate. It is never a good deal for the borrower, unless of course that is all you can get and you must have the loan! Add-on interest is calculated as if you borrowed the entire amount of money, kept it for the duration of the loan period and paid the entire balance back as one balloon payment on last date due. This final number is divided by the number of months to compute a monthly payment. The monthly payment m is calculated from the following formula:

$$\boxed{m = PV(1+rn)/n} \qquad \text{(Add-on Interest)}$$

where PV is the amount borrowed, r is the monthly interest rate, and n is the number of months of the loan. Add-on interest could also be interpreted as an **interest only loan** with a balloon payment equal to the principal at the end of the loan.

Amortized Interest

With amortized interest you only pay interest on the unpaid balance of your loan each month. The formula for the monthly payment M is much more complicated than the formula for add-on interest, and it is usually done with tables, a calculator, or a computer. We will develop the formula here mainly because it gives insight into how to generate amortization tables in Excel. We will use PV to represent the original loan balance and B_i to represent the loan balance at period i. Note that the loan balance is really a future value, FV, which at the end of the loan period will be equal to zero

$$
\begin{aligned}
B_0 &= PV \\
B_1 &= PV - (m - PVr) \\
&= PV - m + PVr \\
&= PV(1+r) - m
\end{aligned}
$$

$$
\begin{aligned}
B_2 &= B_1(1+r) - m \\
&= (PV(1+r) - m)(1+r) - m \\
&= PV(1+r)^2 - m(1+r) - m
\end{aligned}
$$

$$
\begin{aligned}
B_3 &= B_2(1+r) - m \\
&= PV(1+r)^2 - m)(1+r) \\
&= PV(1+r)^3 - m(1+r)^2 - m(1+r) - m
\end{aligned}
$$

Note that the iterative formula $B_n = B_{n-1}(1+r) - m$ is conveniently used in Excel to generate an amortization table. At this point the general form becomes clear and we can write

$$B_n = PV(1+r)^n - m\sum_{i=0}^{n-1}(1+r)^i = 0$$

This whole thing equals zero because the loan has a balance of zero at the end of the loan. This allows us to write

$$PV(1+r)^n = m\sum_{i=0}^{n-1}(1+r)^i$$

Now, pause and study the left and right sides of this equation. The left side is the FV of the PV compounded out for n periods. The right hand side is the same expression used to generate the formula for the future value of a monthly savings amount m. We can now rewrite the formula as

$$\boxed{PV(1+r)^n = \frac{m\left[(1+r)^n - 1\right]}{r}}$$

FV of loan amount = FV of loan payments

This allows us to look at a loan payment in an entirely new way. If we take the amount of money we borrowed and put it in the bank at r% for n years, we would get some future value given by the left side of the above equation. Now, if we find a payment, m, which could save to reach that same amount we would find the payment we needed to pay the loan off. There is a beautiful symmetry between loan payments and savings plans. The final format for the monthly loan payment m on a loan of amount PV is

$$\boxed{m = \frac{PVr(1+r)^n}{(1+r)^n - 1}} \quad \text{or} \quad \boxed{PV = \frac{m\left[(1+r)^n - 1\right]}{r(1+r)^n}} \qquad \text{(Loan Payment)}$$

This really isn't as bad as it looks since the term $(1+r)^n$ is used twice and only need be calculated once. In the above formula, PV is the loan amount, r is the monthly interest rate, and n is the number of months. An alternative format, which you may find easier to calculate, is

$$\boxed{m = \frac{PVr}{1 - \frac{1}{(1+r)^n}}}$$

Symmetry Between Savings and Loans

We will now explore in more detail the symmetry between savings annuities and loans. First let's write the formula to save $1000 dollars in the future

$$\boxed{FV = s\frac{(1+r)^n - 1}{r} = 1000}$$

and now the formula to borrow $1000

$$\boxed{PV = \frac{m\left[(1+r) \;\; - 1\right]}{r(1+r)^n} = 1000}$$

Since they are both equal to 1000, we can set them equal and solve for m in terms of s. After some algebra, we get the fascinating result that

$$\boxed{m = s(1+r)^n}$$

That is, the monthly payment to pay off a loan is equal to the monthly payment to save that same amount multiplied by the IAF of $(1+r)^n$. Now let's consider some numbers. Assume we want to save $1000 in a year at 6% and that we want to pay off a loan of $1000 at 6% in a year. The savings payment is $81.07 and the loan payment is $86.07. Notice that

$$\$86.07 = \$81.07\ (1+.06/12)12$$

Perhaps this is obvious after you see the result, but like many "obvious" results, it takes a little reflection to really get it. At any rate, it is a very insightful result.

There is another fascinating piece of symmetry between savings and loans as illustrated in the following table. Under the columns associated with the monthly savings plan, the Diff column is the change in the monthly balance between successive months. Notice that this amount is exactly equal to the amount of principal paid on a loan of the same amount, which in this case is $1000. Notice that the savings plan begins at zero and grows to $1000, while the loan begins at $1000 and grows to zero. You can think of a loan as savings plan going backwards.

Monthly Savings Plan Payment = $81.07			Loan Amortization Payment = $86.07		
Month	Balance	Diff	Interest	Principal	Balance
1	$81.07	$81.07	$5.00	$81.07	$918.93
2	$162.54	$81.47	$4.59	$81.47	$837.46
3	$244.42	$81.88	$4.19	$81.88	$755.58
4	$326.71	$82.29	$3.78	$82.29	$673.29
5	$409.41	$82.70	$3.37	$82.70	$590.59
6	$492.52	$83.11	$2.95	$83.11	$507.48
7	$576.05	$83.53	$2.54	$83.53	$423.95
8	$659.99	$83.95	$2.12	$83.95	$340.01
9	$744.36	$84.37	$1.70	$84.37	$255.64
10	$829.15	$84.79	$1.28	$84.79	$170.85
11	$914.36	$85.21	$0.85	$85.21	$85.64
12	$1,000.00	$85.64	$0.43	$85.64	($0.00)

The savings balance after period i is

$$FV_i = s\frac{(1+r)^i - 1}{r}$$

and the balance after period i-1 is

$$FV_{i-1} = s\frac{(1+r)^{i-1} - 1}{r}$$

If we subtract the two and plow through some algebra, we get the result that

$$FV_i - FV_{i-1} = s(1+r)^{i-1}$$

Consider month 6 in the above table: 83.11 = 81.07 $(1+.06/12)^5$. Now, let's derive an expression for the principal payment for a loan at period i, which can be expressed as

Principal payment = monthly payment – interest owed

Interest owed = previous loan balance * interest rate

We now need an expression for previous loan balance, which can be written as

$$\boxed{L(1+r)^{i-1} - \frac{m\left[(1+r)^{i-1} - 1\right]}{r}} \qquad \text{(Loan Balance Period i-1)}$$

The principal payment for period i is now equal to the somewhat tedious formula

$$P_i = m - \left[L(1+r)^{i-1} - \frac{m\left[(1+r)^{i-1} - 1\right]}{r} \right] r$$

where

$$L = \frac{m\left[(1+r)^n - 1\right]}{r(1+r)^n}$$

If you substitute for L and plow through another pile of algebra, you will arrive at the amazingly simple result that

$$\boxed{P_i = \frac{m}{(1+r)^{n-(i-1)}}}$$

Now if we substitute $m = s(1+r)^n$, we get

$$P_i = s(1+r)^{i-1} = FV_i - FV_{i-1}$$

Annual Percentage Rate (APR)

APR is the interest rate that would amortize a loan at a given payment. It is used to compare add-on rates to amortized rates. When compared to add-on rates, APR is the amortized rate that would yield the same monthly payment as computed from the add-on rate. Note that the add-on rate is also called the nominal rate because it is the rate "in name only." The true rate is the amortization rate or the APR. This process can be seen by comparing the following two tables for a loan of $1000.

Table 1.1 Monthly payments for add-on interest

	12	24	36	48	60
1%	84.17	42.50	28.61	21.67	17.50
2%	85.00	43.33	29.44	22.50	18.33
3%	85.83	44.17	30.28	23.33	19.17
4%	86.67	45.00	31.11	24.17	20.00
5%	87.50	45.83	31.94	25.00	20.83
6%	88.33	46.67	32.78	25.83	21.67
7%	89.17	47.50	33.61	26.67	22.50
8%	90.00	48.33	34.44	27.50	23.33
9%	90.83	49.17	35.28	28.33	24.17
10%	91.67	50.00	36.11	29.17	25.00
11%	92.50	50.83	36.94	30.00	25.83
12%	93.33	51.67	37.78	30.83	26.67
13%	94.17	52.50	38.61	31.67	27.50
14%	95.00	53.33	39.44	32.50	28.33
15%	95.83	54.17	40.28	33.33	29.17

Table 1.2 Monthly payments of amortization interest

	12	24	36	48	60
10%	$87.92	$46.14	$32.27	$25.36	$21.25
11%	$88.38	$46.61	$32.74	$25.85	$21.74
12%	$88.85	$47.07	$33.21	$26.33	$22.24
13%	$89.32	$47.54	$33.69	$26.83	$22.75
14%	$89.79	$48.01	$34.18	$27.33	$23.27
15%	$90.26	$48.49	$34.67	$27.83	$23.79
16%	$90.73	$48.96	$35.16	$28.34	$24.32
17%	$91.20	$49.44	$35.65	$28.86	$24.85
18%	$91.68	$49.92	$36.15	$29.37	$25.39
19%	$92.16	$50.41	$36.66	$29.90	$25.94
20%	$92.63	$50.90	$37.16	$30.43	$26.49
21%	$93.11	$51.39	$37.68	$30.97	$27.05
22%	$93.59	$51.88	$38.19	$31.51	$27.62
23%	$94.08	$52.37	$38.71	$32.05	$28.19
24%	$94.56	$52.87	$39.23	$32.60	$28.77

Note that the monthly payment for a $1000 loan at 8% for 12 months is $90. This same payment in the amortization table corresponds to an interest rate of 14.5%. Therefore the APR for an add-on rate of 8% for 12 months is 14.5%. There is no algebraic equation for the above relationship, so it is best solved with the use of tables.

Average Daily Balance (ADB)

The average daily balance (ADB) calculation is used by credit card companies to compute the amount of interest due during the billing period. ADB is a weighted average of the credit card balance. This can be written as

$$ADB = (B_1D_1 + B_2D_2 + B_3D_3 + \cdots + B_nD_n) / (D_1 + D_2 + D_3 + \cdots + D_n)$$

Where Bi is the balance that existed for D_i days. In summation notation, this becomes

$$\boxed{ABD = \frac{\sum_{i=1}^{n} B_i D_i}{\sum_{i=1}^{n} D_i}}$$

Note that the sum of all the days should equal the duration of the billing period, usually 30 or 31 days. The actual interest charge is computed by multiplying the ABD by the daily interest rate times the number of days in the billing period. If the annual interest rate is r, then the daily interest rate is r/365. The finance charge can be written as

Finance Charge = ABD × r/365 × days

The new balance becomes the ending balance plus the finance charge.